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ON THE STABILITY PROBLEM FOR THE LAGRANGE SOLUTIONS OF THE RESTRICTED THREE-BODY PROBLEM

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We construct an example of a Hamiltonian system with three degrees of freedom whose equilibrium position is, according to Arnol'd's theorem, stable for the majority of initial conditions but is unstable in the sense of Liapunov. We investigate the problem of the formal stability of triangular Lagrange solutions.

1. We analyze the motion of a body P with infinitesimal mass under the action of the Newtonian attraction of two bodies S and J with masses $1-\mu$ and μ. We assume that bodies S and J move on circular orbits relative to their center of mass. We investigate the stability of the motion of body P under which it forms with bodies S and J an equilateral triangle, assuming that in its own perturbed motion it can leave the rotation plane of bodies S and J. The stated problem was examined in [1]. The investigation was based on the transformation of the Hamilton function to normal form with a subsequent application of Chetaev's instability theorem [2] and of Arnol'd's theorem on the stability of multidimensional Hamiltonian systems [3]. It has been proved that in the region of stability in the linear approximation

$$0 < \mu < \mu^* = 0.0385208 \tag{1.1}$$

the triangular motion in the three-body problem is stable for the majority (in the Lebesgue measure sense) of initial conditions for all μ except the two values $\mu_1 = 0.0242938$ and $\mu_2 = 0.0135160$ for which Liapunov instability takes place.

Let ω_1 and ω_2 $(\omega_1 > \omega_2)$ be the frequencies of plane oscillations of body P in the neighborhood of the triangular motion which satisfy the equation

$$\omega^4 - \omega^2 + {}^{27}/_4\, \mu (1-\mu) = 0$$

It was shown in [1] that if $\mu \neq \mu_1$ and $\mu \neq \mu_2$, then by a suitable choice of the coordinate system the Hamilton function describing the motion in the neighborhood of the triangular solution can be represented in the form

$$H = L + N + H^* (q_i, p_i) \tag{1.2}$$

Here

$$L = \omega_1 r_1 - \omega_2 r_2 + r_3 \tag{1.3}$$

$$N = c_{200} r_1^2 + c_{110} r_1 r_2 + c_{101} r_1 r_3 + c_{020} r_2^2 + c_{011} r_2 r_3 + c_{002} r_3^2$$

$$q_i = \sqrt{2r_i} \sin \varphi_i, \qquad p_i = \sqrt{2r_i} \cos \varphi_i \qquad (i = 1, 2, 3)$$

$$c_{200} = \frac{\omega_2^2 (124\omega_1^4 - 696\omega_1^2 + 81)}{144\,(1-2\omega_1^2)^2 (1-5\omega_1^2)}$$

$$c_{110} = -\frac{\omega_1\omega_2(64\omega_1^2\omega_2^2+43)}{6(1-2\omega_1^2)(1-2\omega_2^2)(1-5\omega_1^2)(1-5\omega_2^2)}$$

$$c_{101} = -\frac{8\omega_1\omega_2^2}{3(1-2\omega_1^2)(4-\omega_1^2)}, \qquad c_{020} = \frac{\omega_1^2(124\omega_2^4-696\omega_2^2+81)}{144(1-2\omega_2^2)^2(1-5\omega_2^2)}$$

$$c_{011} = \frac{8\omega_2\omega_1^2}{3(1-2\omega_2^2)(4-\omega_2^2)}, \qquad c_{002} = -\frac{\omega_1^2\omega_2^2}{3(4-\omega_1^2)(4-\omega_2^2)}$$

Function H^* is analytic in the neighborhood of the origin, being of fifth order of smallness relative to q_i, p_i.

In [1] it was proved that when $\mu \neq \mu_1$ and $\mu \neq \mu_2$ the body P perpetually forms a triangle with bodies S and J, nearly equilateral, for the majority of sufficiently small deviations from the triangle's vertex and for sufficiently small relative velocities. And, according to [3], for these initial conditions the motion of the body P is conditionally periodic with frequencies $\Lambda_i = \partial(L+N)/\partial r_i$ $(i = 1, 2, 3)$. Thus, with probability close to unity the triangular solution is stable. But it is not clear what the nature of the motion is for initial conditions corresponding to commensurable (or almost commensurable) frequencies Λ_i. The system being investigated is three-dimensional. Therefore, Liapunov stability does not at all necessarily follow from stability for the majority of initial conditions. In [4] Arnol'd showed that in the case of commensurable frequencies Λ_i a multidimensional Hamilton system can be unstable. Below we construct a very simple example of such a kind specially for the case of the equilibrium position of an autonomous Hamiltonian system with three degrees of freedom.

2. Let the variation with time of the variables r_i, φ_i $(i = 1, 2, 3)$ be described by differential equations with a Hamilton function

$$H = \omega_1 r_1 - \omega_2 r_2 + \omega_3 r_3 + r_1 r_3 - r_1 r_2 + r_2 r_3 + H^*(r_i, \varphi_i) \tag{2.1}$$
$$H^* = r_1 r_2 \sqrt{r_3} \sin(2\varphi_1 + 2\varphi_2 + \varphi_3)$$

while the frequencies ω_i of the linearized system are positive and connected by the resonance relation

$$2\omega_1 - 2\omega_2 + \omega_3 = 0 \tag{2.2}$$

A system with Hamilton function (2.1) has the origin $r_1 = r_2 = r_3 = 0$ as the equilibrium position. When investigating the motion in a small neighborhood of the origin the function H^* can be considered as a perturbation of a system with the Hamilton function $H^\circ = H - H^*$.

We will show that for a majority of sufficiently small initial values r_i the equilibrium position of the perturbed system is stable. According to [3], to do this it is sufficient to verify that the fourth-order determinant

$$D = \det \begin{Vmatrix} \dfrac{\partial^2 H^\circ}{\partial r_i \partial r_j}, & \dfrac{\partial H^\circ}{\partial r_i} \\ \dfrac{\partial H^\circ}{\partial r_j}, & 0 \end{Vmatrix}$$

is nonzero for $r_1 = r_2 = r_3 = 0$. Expanding this determinant, we obtain

$$D = \omega_1^2 + \omega_2^2 + \omega_3^2 + 2\omega_1\omega_2 + 2\omega_1\omega_3 - 2\omega_2\omega_3$$

By substituting in the place of ω_3 its expression in terms of ω_1 and ω_2 from relation

(2.2), we have $D=(\omega_1+\omega_2)^2\neq 0$. Thus, the equilibrium position is stable for a majority of initial conditions, and the motion in a small neighborhood of the origin is conditionally periodic with frequencies

$$\Lambda_1=\omega_1+r_3-r_2,\quad \Lambda_2=-\omega_2-r_1+r_3,\quad \Lambda_3=\omega_3+r_2+r_3$$

We now show that the equilibrium position is Liapunov unstable. The proof is based on seeking an unboundedly increasing particular solution of the system of differential equations, corresponding to the Hamilton function (2.1)

$$\begin{aligned}&dr_1/dt=dr_2/dt=2dr_3/dt=-2r_1r_2\sqrt{r_3}\cos\psi\\&d\varphi_1/dt=\Lambda_1+r_2\sqrt{r_3}\sin\psi,\qquad d\varphi_2/dt=\Lambda_2+r_1\sqrt{r_3}\sin\psi,\\&d\varphi_3/dt=\Lambda_3+{}^1/_2\, r_1r_2r_3^{-1/2}\sin\psi\\&\psi=2\varphi_1+2\varphi_2+\varphi_3\end{aligned}\tag{2.3}$$

It can be verified that the system of equations (2.3) admits of particular solutions for which $r_1=r_2=2r_3$. Here r_3 and ψ satisfy the following system of differential equations:

$$dr_3/dt=-4r_3^{5/2}\cos\psi,\quad d\psi/dt=10r_3^{3/2}\sin\psi\tag{2.4}$$

For a solution chosen in this fashion the frequencies Λ_i satisfy the resonance relation $2\Lambda_1+2\Lambda_2+\Lambda_3=0$. From system (2.4) we obtain the following particular solution:

$$r_3(t)=r_3(0)\,[1-6r_3^{3/2}(0)\,t]^{-2/3},\quad \psi=\pi$$

From the solution obtained we see that initial conditions arbitrarily close to the origin exist, for which the trajectory of system (2.3) goes arbitrarily far from the origin in course of time; this requires a time of the order of $r_3^{-3/2}(0)$.

3. We can approach the problem of the stability of the triangular solutions also from a formal point of view. The equilibrium position $q_i=p_i=0$ is said to be formally stable [5] if there exists a formal (i.e. possibly, divergent) power series $G=G_n+G_{n+1}+\ldots$, which serves as a sign-definite integral. This signifies that the coefficents of the series

$$\sum_{i=1}^{3} G_{q_i}H_{p_i}-G_{p_i}H_{q_i}+G_t$$

are identically zero, while the function $G_n\geqslant 0$ and vanishes only at the origin $q_i=p_i=0$.

We construct the formal integral for a system with Hamilton function (1.2). In the Hamilton function we can normalize terms of the fifth, sixth, etc. orders by means of the Birkhoff transformation [6]. If μ satisfies condition (1.1) of stability in the linear approximation and does not equal μ_1 and μ_2, then the Hamilton function (1.2), normalized in all orders, is written as

$$H=L+N+R\,(r_i,\varphi_i)\tag{3.1}$$

where L and N are defined by equalities (1.3), and the formal series R commences with terms of the fifth order in $r_i^{1/2}$. The angular variables φ_i are contained in R as combinations

$$k_1\varphi_1+k_2\varphi_2+k_3\varphi_3\tag{3.2}$$

where the k_i are integers for which the equality

$$k_1\omega_1-k_2\omega_2+k_3=0\qquad (|k_1|+|k_2|+|k_3|\geqslant 5))\tag{3.3}$$

is fulfilled.

A system with the Hamiltonian (3.1) has the trivial integral $H =$ const, since H does not depend on time. Furthermore, taking (3.2) and (3.3) into account, it is not difficult to verify that the expression for L also is an integral.

We set up the formal integral G in the form

$$G = L^4 + (H - L)^2 \tag{3.4}$$

In the expansion $G = G_8 + G_9 + \ldots$ the function G_8 has the form $G_8 = L^4 + N^2$. Both terms on the right-hand side of this equality are nonnegative. Therefore, function G_8 is sign-definite in the neighborhood of the origin if in the region $r_1 \geqslant 0$, $r_2 \geqslant 0$, $r_3 \geqslant 0$ the system of equations

$$L = 0, \quad N = 0 \tag{3.5}$$

has only the trivial solution $r_1 = r_2 = r_3 = 0$.

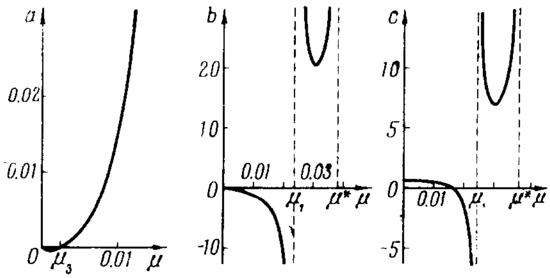

Fig. 1

We investigate the system of equations (3.5). From the first equation $L = 0$ we find an expression for r_3 in terms of r_1 and r_2 and we substitute it into the second equation. Then system (3.5) is rewritten as

$$r_3 = \omega_2 r_2 - \omega_1 r_1, \quad a r_1^2 + b r_1 r_2 + c r_2^2 = 0 \tag{3.6}$$

Here

$$a = c_{200} - c_{101}\omega_1 + c_{002}\omega_1^2, \quad b = c_{110} + c_{101}\omega_2 - c_{011}\omega_1 - 2\,c_{002}\omega_1\omega_2$$

$$c = c_{020} + c_{011}\omega_2 + c_{002}\omega_2^2$$

The graphs of coefficients a, b, c are shown in Fig. 1. Coefficient a vanishes for the value $\mu = \mu_3 = 0.00278$ For this value of μ the coefficients of the system of equations (3.6) are:

$$\omega_1 = 0.99042, \quad \omega_2 = 0.13811, \quad b = -0.39924, \quad c = 0.56461$$

and system (3.6) has the following solutions:

1) r_1 is arbitrary, $r_2 = 0$, $r_3 = -\omega_1 r_1$
2) $r_1 = 1{,}4142\ r_2$, r_2 is arbitrary, $r_3 = -1.26253\ r_2$.

These solutions do not lie inside the region $r_1 \geqslant 0$, $r_2 \geqslant 0$, $r_3 \geqslant 0$. Therefore, when $\mu = \mu_3$, the system (3.6) has only the trivial solution in this region.

For values of μ not equal to μ_i $(i = 1, 2, 3)$ and lying in the interval (1.1), the solutions can be written as: $r_1 = \alpha_j r_2$, $r_3 = \beta_j r_2$, r_2 is arbitrary $(j = 1, 2)$, $\alpha_1 = (-b + \sqrt{b^2 - 4ac})\,/\,2a$, $\alpha_2 = (-b - \sqrt{b^2 - 4ac})\,/\,2a$, $\beta_j = \omega_2 - \alpha_j\omega_1$.

The system (3.6) has a solution in the region $r_1 \geqslant 0$, $r_2 \geqslant 0$, $r_3 \geqslant 0$, if and only if the quantities $b^2 - 4ac, \alpha_j, \beta_j$ are simultaneously nonnegative. Calculations show that $b^2 - 4ac > 0$ for all μ from the interval (1.1), α_1 and β_1 always have opposite signs, and the quantities α_2 and β_2 are simultaneously positive only when the inequalities

$$0.010913... < \mu < 0.016376... \tag{3.7}$$

are fulfilled.

Thus, the formal integral (3.4) is sign-definite for all μ from the interval (1.1) except for those values which lie in interval (3.7) and, of course, for the value $\mu = \mu_1$ which is excluded from consideration right from the very start. However, the value $\mu = \mu_2$ falls into interval (3.7). Therefore, the result obtained can be formulated as follows: the triangular Lagrange solutions of the restricted three-body problem in the three-dimensional circular case are formally stable for all μ in the region of stability in the linear approximation except for $\mu = \mu_1$ and, possibly, for values of μ lying in the intervals

$$0{,}010913... < \mu < \mu_2, \quad \mu_2 < \mu < 0{,}016376... \tag{3.8}$$

We should note that the value $\mu = 0.00095388$, corresponding to the Sun-Jupiter system, does not fall into intervals (3.8), while the value $\mu = 0{,}0121506$ for the Earth-Moon system belongs to the first of these intervals.

Formal stability allows us to assert that Liapunov instability does not appear if in the expansion of the Hamilton function we take into account terms of arbitrarily high (but finite) orders. But if trajectories exist on which body P leaves the triangle's vertex, the motion on them takes place very slowly.

In conclusion we note that the investigation of formal stability for values of μ from intervals (3.8) could be carried out by applying the criterion obtained by Briuno [7].

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